Int. J. Solids Structures Vol. 30, No. 17, pp. 2357–2378, 1993
Printed in Great Britain

0020-7683/93 \$6.00+ .00
Pergamon Press Ltd

FRACTURE MECHANICS OF ORTHOTROPIC LAMINATED PLATES—I. THE THROUGH CRACK PROBLEM

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(*Received* 12 *March* 1992; *in revised form* 28 *September* 1992)

Abstract—In this paper a laminated plate which consists of an arbitrary number of dissimilar orthotropic layers under membrane and bending loads is considered. The plate is assumed to have a through crack. The problem is formulated by using Mindlin's first order shear deformation theory and a higher order theory due to Reddy. The sample solutions obtained from these two theories, from the classical theory, and from Reissner's stress-based first order theory show that the stress intensity factors given by the classical theory are highly inaccurate and the three transverse shear deformation theories give roughly the same results. Because of the nonhomogeneous nature of the medium under consideration, the simpler of the two displacement-based shear deformation theories was adopted to carry out the solution of the crack problem in the laminated plates. The main objective of the study is to investigate the influence of the material orthotropy, various stiffness ratios, and relative dimensions concerning the crack size and laminae thicknesses on the stress intensity factors. Only the mode I crack problem is considered. It is shown that if the laminate has a material symmetry with respect to the midplane of the plate, then the membrane and bending solutions of the problem are fully uncoupled. Also, in laminates which consist of bonded dissimilar isotropic layers having the same Poisson's ratio, the membrane component of the crack problem can again be separated and can be treated as a generalized plane stress problem. Otherwise the problem is coupled. Examples are given for both symmetric and nonsymmetric laminates.

1. INTRODUCTION

The question of primary concern in this series of papers will be the fracture mechanics of laminated materials under a combination of membrane and bending loads. With the application to composite laminates and microelectronic devices in mind, it will be assumed that (a) the medium consists of perfectly bonded orthotropic layers of constant thickness, (b) the in-plane dimensions of the composite medium are considerably greater than its thickness, (c) the rectangular coordinates x, y and z are the principal directions of orthotropy for all layers, z being in the thickness direction, (d) the fracture is confined to a plane of orthotropy perpendicular to the plate (in this case the yz plane), and (e) the external loads are symmetric with respect to the plane of the crack. From a viewpoint of fracture, fatigue and corrosion, in orthotropic materials the planes of orthotropy are generally also the planes of relatively weak fracture resistance. Therefore, in many cases the fracture process may initiate at the surface of the composite laminate in a plane of material symmetry and grow subcritically in both the thickness and the length directions. Upon reaching an interface the crack may be arrested, may grow only in the length or the thickness direction or in both directions, or a delamination crack may initiate and grow along the interface. If the delamination does not take place, then the crack may eventually grow through the entire plate thickness. The progress of the crack front can be monitored analytically by following the procedure described by Joseph and Erdogan (1989) provided the appropriate subcritical crack growth characterization of the material and a technique of calculating the stress intensity factor for a part-through crack front of arbitrary profile are available.

The three-dimensional surface crack problem as stated is analytically intractable. However, it has been shown that representing the medium by a "plate", using the concept of the "line spring model" [e.g. Rice and Levy (1972)], and by using a refined plate theory, it is possible to obtain a solution to the surface crack problem that compares rather well with the existing finite element results given, for example, by Newman and Raju (1979) [see Joseph and Erdogan (1989) for various comparisons]. In this study the application of the plate theory-line spring method will be extended to orthotropic laminated plates having a surface crack. The application of the method requires the formulation of the laminated plate containing a through crack under arbitrary membrane and bending loads and the solution of the corresponding plane strain problem for the composite medium with an edge crack. These two problems will, therefore, be considered separately to obtain the information needed for the solution of the surface crack problem, which will then be described in a third article.

In this first article the primary interest is in the solution of the through crack problem. Some of the factors relevant to the study are the material orthotropy, stacking sequence of the laminae and the particular plate theory used in the formulation. The classical plate theory employing the Kirchhoff hypothesis is described by Lekhnitskii (1968). An improvement of the theory including the bending-membrane coupling in unsymmetrically stacked laminates is discussed by Reissner and Stavsky (1961). Various transverse shear deformation theories including those introduced by Reissner (1945) and Mindlin (1951) are reviewed by Reddy (1989). In addition to being necessary for the formulation of a surface crack problem, the solution of the through crack problem may be useful in studying fracture mechanics of relatively thin laminates in which the effect of bending and membrane-bending coupling may not be negligible. It should also be remarked that other than for isotropic or orthotropic homogeneous plates, no numerical or analytical studies seem to exist which describe the solution of a laminated plate containing a through crack and subjected to remote bending.

2. FORMULATION OF LAMINATED PLATES

In formulating the bending problems in plates by using a first order transverse shear theory one may start with some initial assumptions regarding the thickness distribution of either in-plane stresses or in-plane displacements. Generally the former approach is associated with Reissner and the latter with Mindlin. In both theories the in-plane stress and displacement components turn out to be linear in the thickness coordinate z. Both theories are "approximate" in that they do not satisfy all of the equations of continuum elasticity. The inconsistency in the Reissner's theory is in the distribution of transverse displacement (Reissner, 1945), whereas Mindlin's theory does not satisfy the boundary conditions on the plate surfaces (Mindlin, 1951). In the problem under consideration because of the nonhomogeneous nature of the medium it is much more convenient to use a displacement-based theory. Even though in the main problem of interest the elastic properties of the plate are piecewise constant in the thickness direction, the formulation used is applicable to general nonhomogeneous plates in which the elastic moduli may be arbitrary functions of z.

The description of the general laminated plate theory may be found in Yang *et al.* (1966) where Mindlin's theory for homogeneous plates (Mindlin, 1951), is extended to heterogeneous plates and plates which consist of bonded anisotropic layers. In the model the displacement field is assumed to have the form:

$$\begin{aligned} u(x,y,z) &= u_0(x,y)+z\psi_x(x,y), \\ v(x,y,z) &= v_0(x,y)+z\psi_y(x,y), \\ w(x,y,z) &= w(x,y), \quad 0<z<h, \end{aligned} \tag{1}$$

where w is assumed to be independent of z and u_0 and v_0 are the in-plane displacements at the reference plane [midplane $z = h/2$ for the symmetrically laminated plates and $z = 0$ or the neutral plane $z = c_0$ for nonsymmetric plates, $z = 0$ and $z = h$ being the plate boundaries (Fig. 1)]. The unknown functions u_0, v_0, w, ψ_x and ψ_y are determined from the following equilibrium equations:

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0, \quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=0,$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0, \quad \frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-Q_y=0,$$

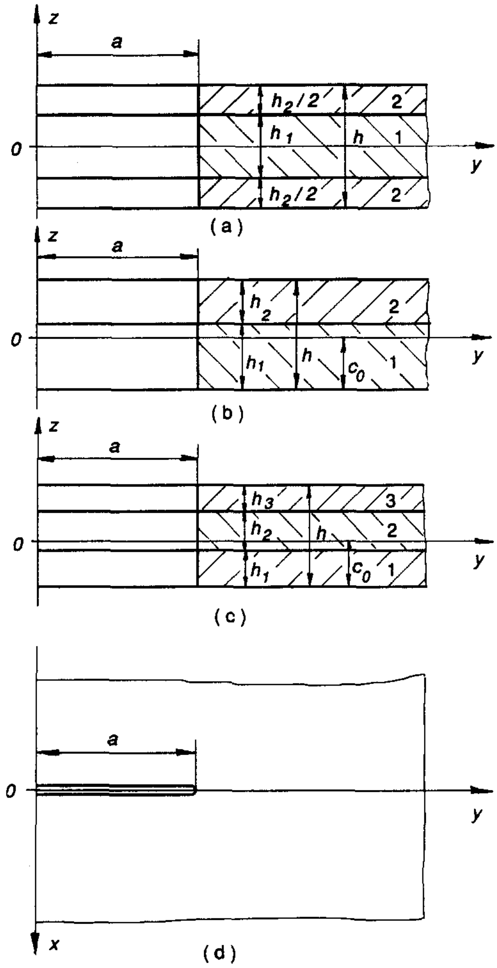

Fig. 1. The geometry and notation for the laminated plate containing a through crack.

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=0, \tag{2}$$

where N_{ij}, Q_i and M_{ij} $(i, j = x, y)$ are the stress and moment resultants in the plate. By defining

$$\frac{\partial u_0}{\partial x}=\varepsilon_{x0},\quad \frac{\partial v_0}{\partial y}=\varepsilon_{y0},\quad \frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}=\gamma_{xy0},$$

$$\frac{\partial \psi_x}{\partial x}=k_x,\quad \frac{\partial \psi_y}{\partial y}=k_y,\quad \frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}=k_{xy}, \tag{3}$$

from (1) and (3) the components of the strain tensor may be expressed as

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}=\varepsilon_{x0}+zk_x,\quad \varepsilon_{yy}=\frac{\partial v}{\partial y}=\varepsilon_{y0}+zk_y,\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\gamma_{xy0}+zk_{xy}, \tag{4}$$

$$\varepsilon_{zz}=\frac{\partial w}{\partial z}=0,\quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\psi_y+\frac{\partial w}{\partial y}=\gamma_{yz0},\quad \gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\psi_x+\frac{\partial w}{\partial x}=\gamma_{xz0}. \tag{5}$$

Also, defining the stress and strain matrices by

$$\sigma = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{yz} \quad \sigma_{xz} \quad \sigma_{xy}]^{\mathrm{T}},$$
$$\varepsilon = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}]^{\mathrm{T}}, \tag{6}$$

the stress–strain relations for an orthotropic medium may be written as

$$\sigma = C\varepsilon, \tag{7}$$

where $C = (C_{ij})$, $(i, j = 1, \ldots, 6)$, is the elasticity matrix. By using (7), for the orthotropic plates the stress and moment resultants may be expressed as

$$N = Ke, \tag{8}$$

$$N = [N_{xx} \quad N_{yy} \quad N_{xy} \quad M_{xx} \quad M_{yy} \quad M_{xy} \quad Q_y \quad Q_x]^{\mathrm{T}}, \tag{9}$$

$$e = [\varepsilon_{x0} \quad \varepsilon_{y0} \quad \varepsilon_{xy0} \quad k_x \quad k_y \quad k_{xy} \quad \gamma_{yz0} \quad \gamma_{xz0}]^{\mathrm{T}}, \tag{10}$$

$$K = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0 \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & 0 & 0 \\ B_{21} & B_{22} & 0 & D_{21} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{55} \end{bmatrix}, \tag{11}$$

$$(A_{ij}, B_{ij}, D_{ij}) = \int_0^h (1, z, z^2) C_{ij}(z) \, \mathrm{d}z, \quad (i,j = 1, 2), \tag{12}$$

$$(A_{66}, B_{66}, D_{66}) = \int_0^h (1, z, z^2) C_{66}(z) \, \mathrm{d}z, \tag{13}$$

$$(A_{44}, A_{55}) = \kappa_0 \int_0^h [C_{44}(z), C_{55}(z)] \, \mathrm{d}z, \tag{14}$$

where κ_0 is a constant. In Reissner's stress-based theory (Reissner, 1945), κ_0 has the value 5/6 for an isotropic plate in whch the distribution of the transverse shear stress through the thickness is parabolic. It is calculated to be $\pi/\sqrt{12}$ by Mindlin through matching the circular frequencies of the first antisymmetric mode of thickness-shear vibrations as given by the plate theory (Mindlin, 1951), and by the exact elasticity theory (Timoshenko, 1937). Similarly, κ_0 is assumed to be 2/3 in considering the transverse vibrations of a homogeneous plate (Uflyand, 1948). However, if one resorts to no artificial means, generally in displacement based shear deformation theories κ_0 turns out to be 1 (Yang *et al.*, 1966). From (12) it may be observed that since $C_{ij} = C_{ji}$, we have $A_{ij} = A_{ji}$, $B_{ij} = B_{ji}$ and $D_{ij} = D_{ji}$.

In this study it is assumed that the plate surfaces $z = 0$ and $z = h$ are traction-free and appropriate moment and stress resultants M_{ij}, N_{ij}, Q_i $(i, j = x, y)$, are prescribed along the boundaries of the plate away from the crack region. Generally, the plate is assumed to consist of n perfectly bonded homogeneous orthotropic layers of constant thickness h_k $(k = 1, \ldots, n)$, $\Sigma h_k = h$. In this case the functions $C_{ij}(z)$ would be piecewise constant. If the midplane of the plate is a plane of material symmetry, then by selecting $z = 0$ at the midplane, from (12) and (13) it may be seen that since $C_{ij}(z) = C_{ij}(-z)$, the partitioned matrix B becomes zero [i.e. $B_{ij} = 0$, $(i, j = 1, 2)$, $B_{66} = 0$] and the "membrane" and "bending" solutions of the problem would be fully uncoupled. In fact, from (2), (3) and (8)–(12) it is seen that the first two equations of (2) giving the unknown functions u_0 and v_0 correspond to the membrane component of the problem. This is a generalized plane stress problem. The remaining three equations in (3) give the unknowns w, ψ_x and ψ_y and constitute the bending component. Clearly, if there is no material symmetry, B_{ij} and B_{66} would not be identically zero and the bending and membrane responses of the plate would always be coupled. In this case, by substituting from (8)–(10) and (3) into (2) we obtain a system of five second order differential equations in terms of five unknown functions u_0, v_0, w, ψ_x and ψ_y.

In the foregoing formulation which is essentially given by Yang *et al.* (1966), since $\varepsilon_{zz} = 0$, it is not possible to make a statement about the distribution of σ_{zz}. From (4), (6) and (7) it follows that σ_{zz} is linear in each layer and does not satisfy the boundary conditions at $z = 0$ and $z = h$. Since the result $\varepsilon_{zz} = 0$ is somewhat unrealistic, one could replace it by the condition that various weighted averages of σ_{zz} in $0 < z < h$ are zero (Whitney and Pagano, 1970), namely

$$\int_0^h \sigma_{zz} w_i(z)\,dz = 0, \quad i = 1, \ldots, 4, \tag{15}$$

where the weight functions w_i are defined below. To follow this alternate approach, we solve ε_{zz} from the third equation of (7) and substitute the result into the first and second. Thus, the first three equations of (7) become

$$\sigma_i = \bar{C}_{i\alpha}\varepsilon_\alpha + \frac{C_{i3}}{C_{33}}\sigma_3, \quad (i = 1, 2, 3), \quad (\alpha = 1, 2), \tag{16}$$

$$\bar{C}_{i\alpha} = C_{i\alpha} - \frac{C_{i3}}{C_{33}} C_{3\alpha}, \quad (i = 1, 2, 3), \quad (\alpha = 1, 2). \tag{17}$$

By using (16) and (17) it may easily be shown that the stiffness matrix K defined by (11) must be replaced by $\bar{K}$ which in turn, is obtained from (11) and (12) by replacing $C_{ij}(z)$ by $\bar{C}_{ij}(z)$, $(i, j = 1, 2)$ and by assuming the weight functions w_i to be

$$w_1(z) = \frac{C_{13}}{C_{33}}, \quad w_2(z) = \frac{C_{23}}{C_{33}}, \quad w_3(z) = \frac{C_{13}}{C_{33}} z, \quad w_4(z) = \frac{C_{23}}{C_{33}} z. \tag{18}$$

From (3)–(7) it may be observed that in the first order shear deformation theory described in the previous section, in each layer the in-plane stresses σ_{xx}, σ_{yy} and σ_{xy} are linear functions in and the transverse shear stresses σ_{xz} and σ_{yz} are independent of the thickness coordinate z. Thus, since σ_{xz} and σ_{yz} are zero on the plate surfaces $z = 0$ and $z = h$ and are constant for $0 < z < h$, the theory violates the local equilibrium conditions at $z = 0$ and $z = h$. To remove such inconsistencies some higher order shear deformation theories have been proposed (Lo *et al.*, 1977; Tiffen and Lowe, 1963). Since these theories involve terms with additional powers of z and, correspondingly, additional unknown functions, they are more complicated and computationally more demanding. The alternate formulation that will be used in solving the crack problem is due to Reddy (1984). This is an extension of the model described by Levinson (1980) for homogeneous isotropic plates to laminated anisotropic plates. The model accounts for the parabolic variation of transverse shear strains without introducing additional unknowns. It is based on the following displacement field:

$$u(x,y,z) = u_0(x,y)+z\left[\psi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_x+\frac{\partial w}{\partial x}\right)\right],$$
$$v(x,y,z) = v_0(x,y)+z\left[\psi_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\psi_y+\frac{\partial w}{\partial y}\right)\right],$$
$$w(x,y,z) = w(x,y). \tag{19}$$

In addition to the quantities defined by (3)–(5) if we define

$$h_x = -\frac{4}{3h^2}\left(\frac{\partial \psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right),\quad h_y = -\frac{4}{3h^2}\left(\frac{\partial \psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right),$$
$$h_{xy} = -\frac{4}{3h^2}\left(\frac{\partial \psi_y}{\partial x}+\frac{\partial \psi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\,\partial y}\right), \tag{20}$$

the strain field may be expressed as

$$\varepsilon_{xx} = \varepsilon_{x0}+zk_x+z^3h_x,\quad \varepsilon_{yy} = \varepsilon_{y0}+zk_y+z^3h_y,\quad \varepsilon_{zz}=0,$$
$$\gamma_{xy} = \gamma_{xy0}+zk_{xy}+z^3h_{xy},$$
$$\gamma_{yz} = \left(1-4\frac{z^2}{h^2}\right)\gamma_{yz0},\quad \gamma_{xz} = \left(1-4\frac{z^2}{h^2}\right)\gamma_{xz0}. \tag{21}$$

It may then be shown that the plate constitutive equations (8) may be replaced by

$$N = Hc, \tag{22}$$

where N is defined by (9) and

$$c = [\varepsilon_{x0}\quad \varepsilon_{y0}\quad \varepsilon_{xy0}\quad k_x\quad k_y\quad k_{xy}\quad h_x\quad h_y\quad h_{xy}\quad \gamma_{yz0}\quad \gamma_{xz0}]^{\mathrm{T}}, \tag{23}$$

$$H = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & E_{11} & E_{12} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & B_{21} & B_{22} & 0 & E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0 & E_{66} & 0 & 0 \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & F_{11} & F_{12} & 0 & 0 & 0 \\ B_{21} & B_{22} & 0 & D_{21} & D_{22} & 0 & F_{21} & F_{22} & 0 & 0 & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0 & F_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_{55} \end{bmatrix}, \tag{24}$$

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij}) = \int_{-h/2}^{h/2}(1,z,z^2,z^3,z^4)C_{ij}\,\mathrm{d}z,\quad (i,j=1,2),$$
$$(A_{66},B_{66},D_{66},E_{66},F_{66}) = \int_{-h/2}^{h/2}(1,z,z^2,z^3,z^4)C_{66}\,\mathrm{d}z,$$
$$(G_{44},G_{55}) = \int_{-h/2}^{h/2}\left(1-\frac{4z^2}{h^2}\right)(C_{44},C_{55})\,\mathrm{d}z. \tag{25}$$

In (24) also it is assumed that $\varepsilon_{zz}=0$. On the other hand if the condition $\varepsilon_{zz}=0$ is replaced by (15), then in (24) C_{ij} $(i, j = 1,2)$ must again be replaced by $\bar{C}_{ij}$ $(i, j = 1,2)$ which is given by (17).

3. FORMULATION OF THE CRACK PROBLEM

The problem of interest is shown in Fig. 1. The laminated plate under consideration consists of an arbitrary number of bonded orthotropic layers and contains a through crack of length $2a$. The total thickness of the plate is h and the coordinate axes x, y, z are assumed to coincide with the principal axes of orthotropy in each layer. The transverse shear deformation theory for laminated heterogeneous plates described in Section 2 will be used to formulate and solve the crack problem. For comparison some limited results obtained from a higher order shear deformation theory and from Reissner's theory will also be provided.

Defining now the displacement matrix by

$$u = (u_i) = [u_0 \quad v_0 \quad \psi_x \quad \psi_y \quad w]^{\mathrm{T}}, \quad (i = 1, \ldots, 5) \tag{26}$$

and substituting from (3)–(5) and (8)–(10) into the plate equilibrium equations (2) we obtain a system of second order partial differential equations of the following form:

$$L_i[u_j(x, y)] = 0, \quad (i, j = 1, \ldots, 5). \tag{27}$$

If we express $u_j(x, y)$ by the following Fourier integrals

$$u_j = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_j(x, \alpha)\, \mathrm{e}^{-i\alpha y}\, \mathrm{d}\alpha, \quad (j = 1, \ldots, 5), \tag{28}$$

from (27) it follows that

$$P\phi_{,xx} + Q\phi_{,x} + R\phi = 0, \tag{29}$$

where

$$\phi = [\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4 \quad \phi_5]^{\mathrm{T}}, \tag{30}$$

and the square matrices P, Q and R are known functions of α and the material constants defined by (12)–(14) [see Wu (1990) for details].

We assume that the applied loads are symmetric with respect to the $x = 0$ plane and through a proper superposition the crack problem is reduced to a local perturbation problem in which the crack surface membrane and moment resultants N_{xx} $(0, y)$ and M_{xx} $(0, y)$ are the only non-zero external loads. Thus, upon determining the eigenvalues s_i and eigenfunctions S_{ij} of the system of differential equations (29), its solution may be expressed as

$$\phi_j = \sum_{k=1}^{5} S_{jk}(\alpha) A_k(\alpha)\, \mathrm{e}^{s_k x}, \quad \mathrm{Re}\,(s_k) < 0, \quad (j, k = 1, \ldots, 5), \tag{31}$$

where the unknown functions A_k are determined by using the boundary conditions at $x = 0$. Because of symmetry the problem is a mode I crack problem and is subjected to the following boundary conditions:

$$N_{xy}(0, y) = 0, \quad M_{xy}(0, y) = 0, \quad Q_x(0, y) = 0, \quad -\infty < y < \infty, \tag{32}$$

$$N_{xx}(0^+, y) = hf_1(y), \quad |y| < a, \quad u_0(0, y) = 0, \quad |y| > a, \tag{33}$$

$$M_{xx}(0^+, y) = h^2/6 \cdot f_2(y), \quad |y| < a, \quad \psi_x(0, y) = 0, \quad |y| > a, \tag{34}$$

where f_1 and f_2 are known functions. The homogeneous equations (32) may be used to eliminate three of the unknown functions $A_j(\alpha)$. The mixed boundary conditions (33) and (34) would then give a system of dual integral equations to determine the remaining two. The mixed boundary value problem may also be reduced to a pair of integral equations in terms of the unknown functions defined by

$$\frac{\partial}{\partial y}u_0(0,y) = g_1(y), \quad \frac{\partial}{\partial y}\psi_x(0,y) = g_2(y), \quad -\infty < y < \infty. \tag{35}$$

Clearly, by substituting from (28) and (31) into (32) and (35), all five unknowns A_j may be expressed in terms of g_1 and g_2. Thus, by observing that $g_1 = 0$, $g_2 = 0$ for $|y| > a$ and by referring to Wu (1990) for details, conditions (33) and (34) may be expressed as

$$\lim_{x\to+0}\int_{-a}^{a}\sum_{1}^{2} g_j(t)\,dt\int_{-\infty}^{\infty} m_{kj}(x,\alpha)\,e^{i\alpha(t-y)}\,d\alpha = f_k(y), \quad k=1,2, \quad |y|<a, \tag{36}$$

where m_{kj} are known functions. The singular behavior of the kernels in (36) may be determined by examining the asymptotic nature of the integrands m_{kj} for $|\alpha| \to \infty$. We note that m_{kj} are functions of s_i and contain the damping terms $\exp(s_i x)$, $(i = 1,\ldots,5)$ where $\mathrm{Re}\, s_i(\alpha) < 0$. The difficulty in this problem, of course, is that the functions $s_i(\alpha)$ are not explicitly known in terms of α. For the purpose of examining the singular behavior of the kernels in (36) all one needs, however, is the asymptotic expressions of $s_i(\alpha)$ for $|\alpha| \to \infty$. Thus, from the characteristic equation (29) for large values of $|\alpha|$ it can be shown that

$$\frac{s_i(\alpha)}{|\alpha|} = -\left(s_0 + \frac{s_{i1}}{\alpha} + \frac{s_{i2}}{\alpha^2} + \cdots\right), \quad (i=1,\ldots,5), \tag{37}$$

where $s_0, s_{i1}, \ldots$ are constants.

Now, by using the relations (37) and separating the asymptotic values m_{kj} for large $|\alpha|$, (36) may be expressed as

$$\sum_{j=1}^{2}\frac{\mu_{kj}}{\pi}\int_{-a}^{a}\frac{g_j(t)}{t-y}\,dt + \sum_{j=1}^{2}\int_{-a}^{a} k_{kj}(y,t)g_j(t)\,dt = f_k(y), \quad (k=1,2), \quad -a<y<a. \tag{38}$$

In (38) μ_{kj} are material constants obtained from the asymptotic analysis and k_{kj} are the Fredholm kernels. From (33)–(35) it may be seen that eqns (38) must be subject to the following single-valuedness conditions:

$$\int_{-a}^{a} g_j(t)\,dt = 0, \quad (j=1,2). \tag{39}$$

If the midplane of the composite plate is a plane of material symmetry, by selecting it as the reference plane it can be shown that in (36) m_{12} and m_{21} are zero giving $\mu_{12} = 0$, $\mu_{21} = 0$, $k_{12} = 0$ and $k_{21} = 0$. In this case the crack problems for the layered plate under membrane and bending loads would be uncoupled.

4. SOLUTION AND THE STRESS INTENSITY FACTORS

Since the dominant kernels of the system of integral equations (38) are of simple Cauchy type, its solution may be expressed as follows:

$$g_j(t) = \frac{G_j(t)}{(a^2-t^2)^{1/2}}, \quad (j = 1,2), \tag{40}$$

where the functions G_1 and G_2 are bounded in $-a \leqslant t \leqslant a$ and non-zero at $t = \mp a$. The singular integral equations may be solved by defining the normalized quantities

$$r = t/a, \quad s = y/a, \quad g_j(t) = \frac{G_j(ar)/a}{\sqrt{1-r^2}}, \quad (j = 1,2) \tag{41}$$

and by letting (Erdogan, 1978)

$$\frac{G_j(ar)}{a} = \sum_{n=0}^{N} a_{jn} T_n(r), \quad (j = 1,2), \tag{42}$$

the unknown coefficients a_{jn} are determined by substituting from (41) and (42) into (38).

An important physical quantity in the problem is the crack opening displacement defined by

$$u(0,y,z) = u_0(0,y) + z\psi_x(0,y), \quad (-a < y < a). \tag{43}$$

From (35), (40) and (42) it may be shown that

$$u_0(0,y) = \int_{-a}^{y} g_1(t)\,\mathrm{d}t = -\sum_{n=1}^{\infty} \frac{1}{n} a_{1n} U_{n-1}\left(\frac{y}{a}\right)\sqrt{a^2-y^2}, \quad |y| < a, \tag{44}$$

$$\psi_x(0,y) = \int_{-a}^{y} g_2(t)\,\mathrm{d}t = -\sum_{n=1}^{\infty} \frac{1}{n} a_{2n} U_{n-1}\left(\frac{y}{a}\right)\sqrt{a^2-y^2}, \quad |y| < a. \tag{45}$$

From the viewpoint of fracture mechanics perhaps the most important quantity of interest is the stress intensity factor, which, in the mode I problem under consideration, is defined by

$$k_1(z) = \lim_{y \to a^+} \sqrt{2(y-a)}\,\sigma_{xx}(0,y,z). \tag{46}$$

If $y = 0$ is not a plane of symmetry, a similar expression holds for the stress intensity factor at $y = -a$. It can be shown that the stress intensity factor may also be obtained from the following alternate expression:

$$k_1(z) = \lim_{y \to a^-} \bar{\mu}\sqrt{2(a-y)}\frac{\partial}{\partial y}u(0,y,z) \tag{47}$$

where $\bar{\mu}$ is a material constant. For isotropic materials $\bar{\mu}$ is given by

$$\bar{\mu} = \frac{4\mu}{1+\kappa}, \tag{48}$$

where μ is the shear modulus, $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stress conditions. In the shear deformation plate theory used, even though the crack opening displacement is continuous in z, because of the discontinuity in material constants the stress intensity factor is expected to be discontinuous across the interfacial planes and $\bar{\mu}$ in (47) must be determined for each orthotropic layer separately. Defining the inverse of the stiffness matrix C given in (7) by

$$C^{-1} = c = (c_{ij}), \quad (i, j = 1, \ldots, 6), \tag{49}$$

it can be shown that (Sih and Liebowitz, 1968),

$$\bar{\mu} = \frac{1}{2}\left(\frac{d_{11}d_{22}}{2}\right)^{-1/2}\left[\left(\frac{d_{11}}{d_{22}}\right)^{1/2} + \frac{2d_{12}+d_{66}}{2d_{22}}\right]^{-1/2}, \tag{50}$$

where

$$d_{11} = c_{11}, \quad d_{22} = c_{22}, \quad d_{12} = c_{12}, \quad d_{66} = c_{66} \tag{51}$$

for generalized plane stress and

$$d_{11} = \frac{c_{11}c_{33}-c_{13}^2}{c_{33}}, \quad d_{22} = \frac{c_{22}c_{33}-c_{23}^2}{c_{33}}, \quad d_{12} = \frac{c_{12}c_{33}-c_{13}c_{23}}{c_{33}}, \quad d_{66} = c_{66} \tag{52}$$

for plane strain conditions.

5. RESULTS AND DISCUSSIONS

The problem considered in this paper is a three-dimensional crack problem in which the plane of the crack intersects the stress free surfaces perpendicularly and at the point of intersection the stress state has a singularity of the form ρ^{-1}, where ρ is a small distance from this point (Benthem, 1977; Benthem, 1980; Bazant and Estensorro, 1979). The results given by Benthem show that for $0 \leqslant \nu \leqslant 0.5$ we have $0.5 \geqslant \alpha \geqslant 0.3318$ for mode I and $0.5 \leqslant \alpha \leqslant 0.6462$ for modes II and III deformation states. Thus, the stress intensity factors defined on the basis of conventional square root singularity along the crack front $0 < z < h$ would have the behavior $|k_1(z)| \to 0$, $|k_2(z)| \to \infty$ and $|k_3(z)| \to \infty$ as $z \to 0$ and $z \to h$ where $z = 0$ and $z = h$ correspond to the plate surfaces. This traction-free surface effect has indeed been clearly demonstrated by full-field finite element analysis for the in-plane loading conditions (Nakamura and Parks, 1988, 1989). For plates under bending even though one would expect to have a boundary layer with relatively smaller thickness than the plates under membrane loading, the basic trends regarding the behavior of the stress intensity factors near and at the plate surfaces should remain unchanged. On the other hand, as shown by Bazant and Estensorro (1979), the singularity α is dependent on the angle between the crack front and the plate surface as well as the Poisson's ratio. Thus, under conditions of linear elastic fracture, that is for crack propagation in brittle solids and for low amplitude subcritical crack growth in most engineering materials, near the surfaces the crack front adjusts itself in such a way that α becomes 0.5. This would result in a slight "tunneling" for mode I and "chevron" effect for modes II and III crack propagation (Joseph and Erdogan, 1989).

In all finite element and other approximate solutions of plates containing part-through or through cracks and subjected to bending that appeared in literature up to now the traction-free surface effect on the stress singularities seems to have been ignored. From a practical view point the consequence of this is that essentially a curved crack front is approximated by a straight line. Considering all other approximations involved and the fact that in most cases the deviation from a straight crack front is very slight, the assumption would not be expected to cause any significant errors. Whether or not explicitly stated, this has been the rationale for the existing finite element and approximate analytical solutions of the through crack problems in plates under bending. In fact the handful of finite element solutions that exist are obtained for isotropic and homogeneous plates (Barsoum, 1976;

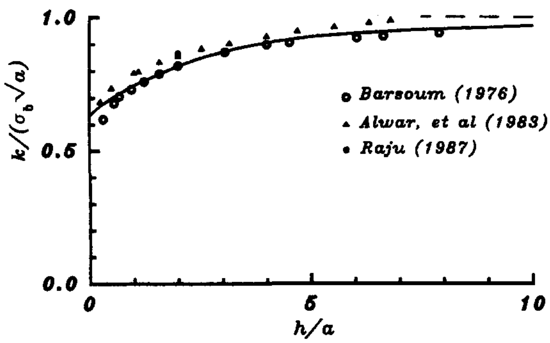

Fig. 2. Comparison of the normalized stress intensity factors given by the finite element method and the Reissner plate theory for a homogeneous isotropic plate with a through crack under cylindrical bending, M_0, $\sigma_b = 6M_0/h^2$, $v = 0.3$.

Alwar and Nambissan, 1983; Raju, 1987). For $v = 0.3$ Fig. 2 shows the comparison of the mode I stress intensity factors $k_1(h/2)$ obtained from these finite element solutions with that given for a homogeneous isotropic Reissner plate under cylindrical bending (Joseph and Erdogan, 1989, 1991). First, it should be remarked that the finite element results shown in Fig. 2 are far from being exact. Secondly, the classical plate theory would give the closed form solution $k_1(z) = (2z/h)(6M_0/h^2)\sqrt{a}$ which is independent of h/a as well as the Poisson's ratio. Thirdly, the agreement shown in the figure indicates that a simple transverse shear theory could be counted upon to give sufficiently reliable results.

Nearly all solutions of the crack problems in plates under cylindrical bending that appeared in literature up to now have been obtained by using either the classical or the Reissner's plate theory. It would, therefore, be worthwhile to compare the results obtained from various plate theories. This comparison is shown in Table 1 for a homogeneous isotropic plate under cylindrical bending. In these and in all subsequent examples given in this paper it is assumed that the weighted averages of σ_{zz} rather than ε_{zz} are zero. That is, it is assumed that the conditions (15) rather than $\varepsilon_{zz} = 0$ are valid and consequently, in (12) and (25a) the material constants $\bar{C}_{ij}$ defined by (17) rather than C_{ij} are used to calculate the coefficients of the plate stiffness matrices K and H. First we note that since the length parameter a/h is lost in formulating the crack problem by using the classical plate theory, the results turn out to be independent of a/h. Secondly, for values of a/h that are large enough for "plate" theories to be valid, from Table 1 it may be seen that the classical plate results are highly inaccurate. Finally, the table shows that the results given by the three shear deformation theories considered are sufficiently close. Thus, one would expect to obtain acceptable results if one uses a relatively simple first order transverse shear theory such as Mindlin's.

To give some idea about the effect of the Poisson's ratio on the stress intensity factors, the results for $v = 0$ and $v = 0.5$ obtained from the Reissner's theory are also included in the table. These results and the asymptotic values of the normalized stress intensity factors

Table 1. Comparison of the normalized stress intensity factor $k_1(h/2)/\sigma_b\sqrt{a}$ for an isotropic homogeneous plate under uniform bending obtained from various shear deformation theories, $\sigma_b = 6M_0/h^2$, $M_0 = M_{xx}(\infty, y)$

a/h	Classical ($\kappa_0 = 0$)	Reissner ($v = 0$) ($\kappa_0 = 5/6$)	Reissner ($v = 0.3$) ($\kappa_0 = 5/6$)	Reissner ($v = 0.5$) ($\kappa_0 = 5/6$)	Mindlin ($v = 0.3$) ($\kappa_0 = 1$)	Reddy ($v = 0.3$) ($\kappa_0 = 1$)
0.05	1.0	0.9851	0.9885	0.9900	0.9869	
0.1	1.0	0.9583	0.9676	0.9717	0.9632	0.9676
0.25	1.0	0.8735	0.8992	0.9111	0.8895	0.8892
0.5	1.0	0.7804	0.8193	0.8383	0.8087	0.8193
1.0	1.0	0.7020	0.7475	0.7707	0.7401	0.7477
2.0	1.0	0.6518	0.6997	0.7247	0.6982	0.7008
$\to\infty$	1.0	0.5774	0.6277	0.6547		

Table 2. The effect of transverse shear correction factor κ_0 on the stress intensity factor in a homogeneous isotropic plate under uniform bending, $\nu = 0.3$, $\sigma_b = 6M_0/h^2$, $M_0 = M_{xx}(\infty, y)$

	κ_0				
a/h	0.0001	5/6	1	10	100
0.05	1.000	0.9885	0.9869	0.9338	0.8141
0.10	1.000	0.9676	0.9632	0.8634	0.7449
0.25	1.000	0.8992	0.8895	0.7610	0.6898
0.5	0.9997	0.8193	0.8087	0.7090	0.6684
1.0	0.9990	0.7475	0.7401	0.6793	

for $(a/h) \to \infty$ are taken from Joseph and Erdogan (1991). Unless the z-dependence is specifically considered, the stress intensity factors given in this paper are those calculated at the plate surface. Needless to say, the bending results given in this paper can be meaningful only if the plate is also under membrane loading of sufficient magnitude so that $k_1(z) > 0$ everywhere along the crack front and there is no contact of the crack surfaces on the compressive side of the plate.

The transverse shear correction factor $\kappa_0 = 5/6$ arises naturally in formulating the plate problem by using the Reissner's theory. In displacement-based first order transverse shear theories κ_0 is introduced artificially. In the higher order shear deformation theory proposed by Reddy, the transverse shear stress distribution is parabolic and satisfy plate surface traction boundary conditions; consequently there is no need for κ_0. In the classical plate theory κ_0 is zero. Table 2 shows the effect of κ_0 on the stress intensity factors $k_1(h/2)$ in an isotropic, homogeneous plate under cylindrical bending. The results are obtained for $\nu = 0.3$ by using Mindlin's shear deformation theory. Note that $k_1(h/2)$ is a decreasing function of κ_0 as well as a/h. In the displacement-based shear deformation theories the derived value of κ_0 is one. Therefore, the remainder of the results given in this paper will be based on the generalized Mindlin's first order shear deformation theory and on the assumption that $\kappa_0 = 1$.

Tables 1 and 2 show the effect of various plate theories and the transverse shear correction factor κ_0 on the stress intensity factor k_1 in homogeneous isotropic plates. To give some idea about the influence of κ_0 on k_1 in laminates which consist of bonded orthotropic or isotropic layers, some limited results are also obtained by using Mindlin's theory with $\kappa_0 = 1$ and $\kappa_0 = 5/6$. The results are shown in Tables 3 and 4. Table 3 shows the stress intensity factors k_1 for a plate which consists of two bonded orthotropic layers under membrane loading or bending moment [see Fig. 1(b)]. The elastic properties of orthotropic materials used for Table 3 are given in Table 5. Note that k_1 is calculated on the surfaces of the plate, that is at $z = h - c_0$ and $z = -c_0$, where c_0 defines the location of the neutral plane. The results for a laminate which consists of three dissimilar isotropic layers are shown in Table 4. The tables show that in a displacement based shear deformation theory assuming the transverse shear correction factor κ_0 to be 1 as it comes out of the analysis or 5/6 to make it conform to the Reissner's stress-based theory does not seem to have any more influence on the stress intensity factors in laminated isotropic or orthotropic plates than they do in isotropic homogeneous plates.

Table 3. The effect of κ_0 on the normalized stress intensity factors $k_1(h-c_0)/k_0$, and $k_1(-c_0)/k_0$ in a plate which consists of two bonded orthotropic layers and is subjected to uniform membrane loading $N_{xx} = N_0$ or cylindrical bending $M_{xx} = M_0$ [Fig. 1(b)]. Layer 1 is Material A, Layer 2 is Material B, $h_1 = h_2 = h/2$, $a/h = 1$, $k_0 = (N_0/h)\sqrt{a}$ for membrane loading, $k_0 = (6M_0/h^2)\sqrt{a}$ for bending

	$k_1(h-c_0)/k_0$		$k_1(-c_0)/k_0$	
	$\kappa_0 = 1$	$\kappa_0 = 5/6$	$\kappa_0 = 1$	$\kappa_0 = 5/6$
Membrane loading	0.8788	0.8775	1.1207	1.1222
Bending moment	0.7921	0.8019	−0.8327	−0.8435

Table 4. The effect of κ_0 on the normalized stress intensity factor $k_1(\bar{z})/k_0$ in a laminate which consists of three different isotropic layers and is subjected to membrane loading $N_{xx} = N_0$ or bending moment $M_{xx} = M_0$ [Fig. 1(c)]. Layer 1: $E_1 = 3$ GPA, $\nu_1 = 0.5$, Layer 2: $E_2 = 1$ GPA, $\nu_2 = 0$, Layer 3: $E_3 = 10$ GPA, $\nu_3 = 0.3$; $h_1/h_2 = 5$, $h_3 = h_1$, $a/h = 1$; $\bar{z}$ is measured from the bottom surface

	Membrane loading		Bending	
$\bar{z}/h$	$\kappa_0 = 1$	$\kappa_0 = 5/6$	$\kappa_0 = 1$	$\kappa_0 = 5/6$
0.0	1.2166	1.2175	−0.7904	−0.7989
0.1429−0	1.1999	1.2006	−0.6187	−0.6253
0.1429+0	0.3995	0.3997	−0.2060	−0.2082
0.8571−0	0.3717	0.3716	0.0799	0.0808
0.8571+0	3.7169	3.7161	0.7933	0.8079
1.0	3.6613	3.6599	1.3712	1.3859

In the examples considered in the remainder of this paper it will be assumed that the laminated plate consists of actual orthotropic laminae, bonded isotropic layers, or layers with hypothetical material properties. The third group of materials were considered for the purpose of studying the effect of specific material constants on the stress intensity factors. The material constants of the orthotropic layers used in the examples are given in Table 5. Both materials are fiber-reinforced graphite-epoxy composites. Note that material B is the same as A except that the medium is rotated 90° about the z-axis relative to the coordinate system.

Figure 1 shows the three types of laminations considered as examples. In all cases the crack is in the yz-plane and the neutral plane is selected as the reference plane $z = 0$. Depending on the relative thicknesses and stiffnesses, the constant c_0 defines the reference plane. The only external load used is either the membrane loading

$$N_{xx}(0, y) = -N_0, \quad -a < y < a, \tag{53}$$

or the bending moment

$$M_{xx}(0, y) = -M_0, \quad -a < y < a, \tag{54}$$

corresponding to uniform neutral plane tensile loading parallel to the x-axis and cylindrical bending about the y-axis.

Figures 3–7 show the normalized stress intensity factor $k_1(h/2)/k_0$ in a three-layer symmetric laminate containing a through crack of length $2a$ [see Fig. 1(a)]. In these figures the normalizing stress intensity factor k_0 is $\sigma_b\sqrt{a}$, where $\sigma_b = 6\text{M}_0/h^2$ is the surface stress $\sigma_{xx}(h/2)$ in a corresponding homogeneous plate of thickness h subjected to uniform bending $M_{xx} = M_0$ away from the crack region. The stress intensity factor k_1 is obtained from (47) at $z = h/2$ by using the appropriate modulus $\bar{\mu}$. In Fig. 3 it is assumed that the core [material

Table 5. The material constants of the orthotropic layers used in the examples. E_i and G_{ij}, $(i, j = x, y, z)$ are in GPA. The materials are fiber-reinforced graphite-epoxy composites

	A	B
E_x	39.0	30.6
E_y	30.6	39.0
E_z	6.4	6.4
G_{xy}	19.7	19.7
G_{yz}	4.5	4.5
G_{xz}	4.5	4.5
ν_{xy}	0.447	0.351
ν_{yz}	0.275	0.275
ν_{xz}	0.275	0.275

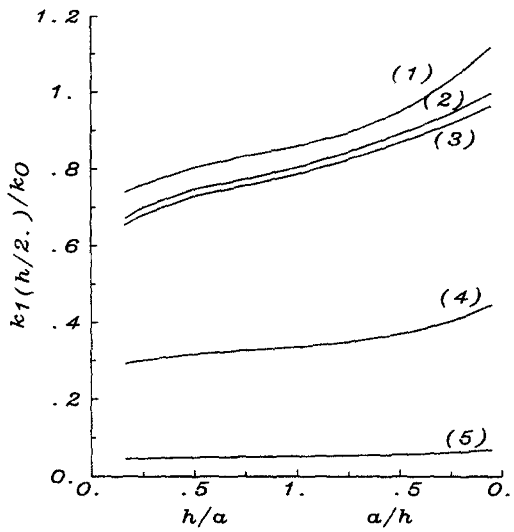

Fig. 3. Normalized stress intensity factor $k_1(h/2)/k_0$ at $z = h/2$ in a three-layer symmetric laminate under cylindrical bending [Fig. 1(a)]. The core is Material A, the surface layers are (1) $E = 390$ GPA, $\nu = 0.3$ (isotr.), (2) Material A, (3) Material B, (4) $E = 3.9$ GPA, $\nu = 0.3$, (5) $E = 0.39$ GPA, $\nu = 0.3$ (labelled in descending order of E_x). $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $h_1 = h_2 = h/2$.

1 in Fig. 1(a)] is Material A shown in Table 5 and the surface layers are any one of the five different materials (1), …, (5). The corresponding stress intensity factors are marked by (1), …, (5) on the figure. Materials (1), …, (5) are either orthotropic (A or B shown in Table 5) or isotropic with the Poisson's ratio $\nu = 0.3$ and the Young's modulus 0.39, 3.9 or 390 GPA. Note that the case (2) in Fig. 3 corresponds to a homogeneous orthotropic plate. In laminates since the stiffer layers would be under greater stress, they would also have greater stress intensity factors than the corresponding homogeneous plates. Also note that, as expected, in all cases k_1 is a monotonically increasing function of h/a.

Figure 4 shows the influence of the modulus ratio E_2/E_1 and h/a on the stress intensity factor at $z = h/2$ in a symmetric laminate that consists of three isotropic layers, where E_1, h_1 and E_2, h_2 relate to the core and the surface layers, respectively, $h_1+h_2 = h$ [Fig. 1(a)], and in all cases shown $\nu = 0.3$. Again, note that the case of $E_2/E_1 = 1$ corresponds to an isotropic plate, stiffer layers have the greater stress intensity factors, the effect of the stiffness ratio E_2/E_1 on k_1 can be very significant, and k_1 is a monotonically increasing function of h/a.

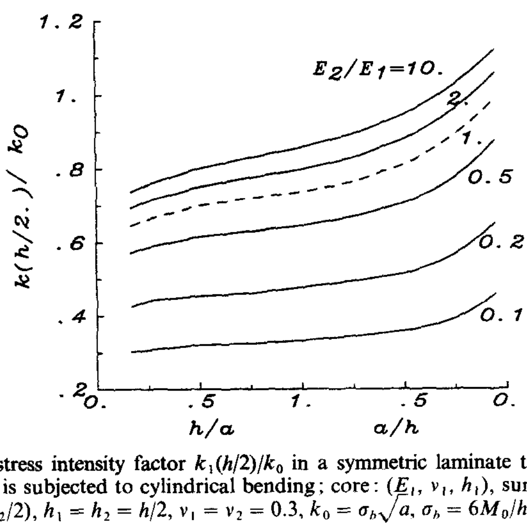

Fig. 4. Normalized stress intensity factor $k_1(h/2)/k_0$ in a symmetric laminate that consists of three isotropic layers and is subjected to cylindrical bending; core: (E_1, ν_1, h_1), surface layers: (E_2, ν_2, $h_2/2$), $h_1 = h_2 = h/2$, $\nu_1 = \nu_2 = 0.3$, $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$.

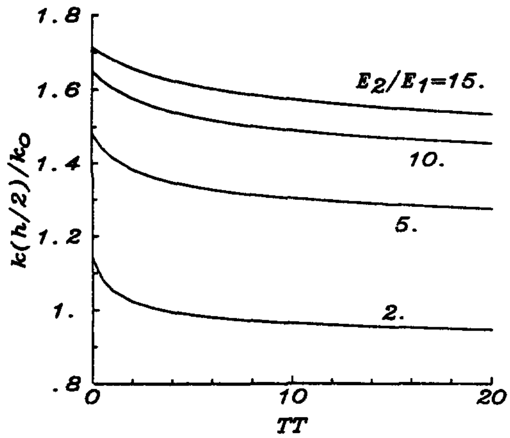

Fig. 5. The effect of stiffness ratios on $k_1(h/2)/k_0$ in a honeycomb plate. $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $h_1/h_2 = 5$, $a/h = 1$, $TT = G_{xz}/G_{xy}$, $G_{yz} = G_{xz}$.

A material of some considerable practical interest is a "honeycomb structure" which can be modeled as a three-layer symmetric laminate with the following features [Fig. 1(a)]:

(a) $h_1 \gg h_2$,
(b) $E_2 \gg E_1$,
(c) Surface layers are isotropic and the core region is orthotropic with transverse shear stiffnesses G_{xz} and G_{yz} being considerably greater than the in-plane shear stiffness G_{xy}.

The stress intensity factor $k_1(h/2)/k_0$ in such a laminate is shown in Figs 5–7. In these examples the isotropic surface layers have the elastic constants E_2 and $\nu_2 = 0.3$. The material properties of the core region are assumed to be $E_x = E_y = E_1$, $G_{xy} = E_1/2(1+\nu_1)$, $\nu_{xy} = \nu_{yx} = 0.3$, and $G_{xz} = G_{yz} = TTG_{xy}$ where the coefficient TT is variable. The results shown are self-explanatory. One may observe that the effect of the shear stiffness ratio TT is not as significant as the effect of E_2/E_1. Also note that in Figs 6 and 7 as $h_1/h_2 \to 0$ (with $h_1+h_2 = h$), in all cases one recovers the isotropic plate result $k_1(h/2)/k_0 = 0.7401$ (Table 1).

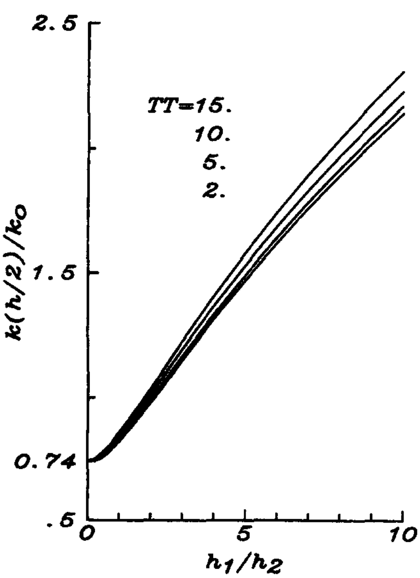

Fig. 6. The effect of TT and h_1/h_2 on $k_1(h/2)/k_0$ in a honeycomb plate. $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $E_1/E_2 = 10$, $a/h = 1$, $TT = G_{xz}/G_{xy}$, $G_{yz} = G_{xz}$.

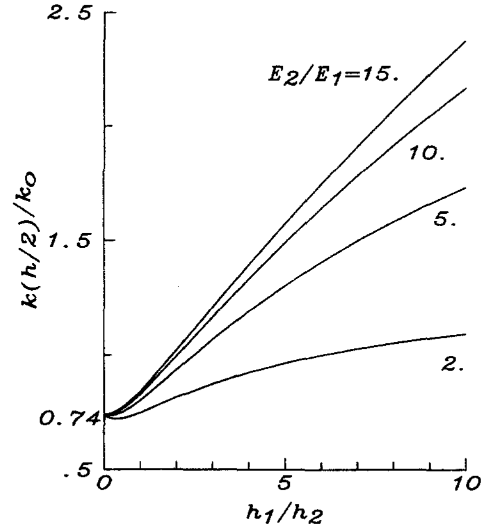

Fig. 7. The effect E_2/E_1 and h_1/h_2 on $k_1(h/2)/k_0$ in a honeycomb plate. $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $TT = G_{xz}/G_{xy} = 10$, $a/h = 1$, $G_{yz} = G_{xz}$.

A second group of examples we consider is concerned with the crack problem for a laminate which consists of only two layers [see Fig. 1(b) for notation and geometry]. Of course, the most significant aspect of these examples is the bending-membrane coupling due to the absence of material symmetry in thickness direction. The mode I stress intensity factor k_1 shown in the figures is calculated at the surface $z = h - c_0$ on the tensile side of the plate where the neutral plane defined by c_0 is assumed to be the reference plane $z = 0$ [Fig. 1(b)]. Figure 8 shows the effect of the modulus ratio E_2/E_1 and a/h on the normalized stress intensity factor. The effect of the thickness ratio h_2/h_1 is shown in Fig. 9 for a plate consisting of two orthotropic layers and subjected to bending. The material properties are given in Table 5. The results concerning the effect of both the stiffness and the thickness ratios conform to the physically expected trends.

As indicated before in a plate that consists of two dissimilar layers, because of the absence of material symmetry in thickness direction the bending and membrane components of the crack solution are always coupled. Thus, since crack opening u_0 and rotation ψ_x are always nonzero regardless of the loading conditions, in addition to the jump discontinuity at the interface, the dependence of the stress intensity factor on z along the crack front would always be linear. This may be seen from the example shown in Fig. 10. The figure

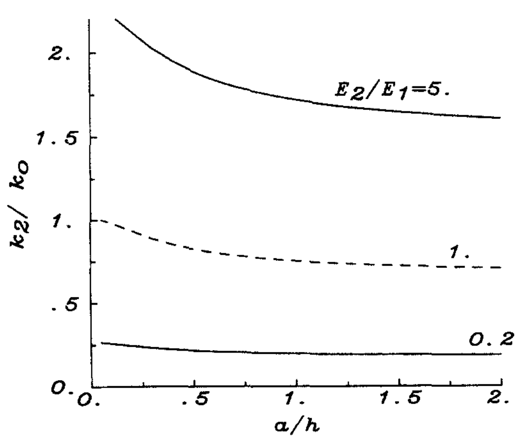

Fig. 8. Normalized stress intensity factor k_1/k_0 in a laminate that consists of two isotropic layers and is subjected to cylindrical bending $M_{xx} = M_0$ away from the crack region. $\nu_1 = \nu_2 = 0.3$, $k_1 = k_1(h - c_0)$, $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $h_2/h_1 = 0.1$.

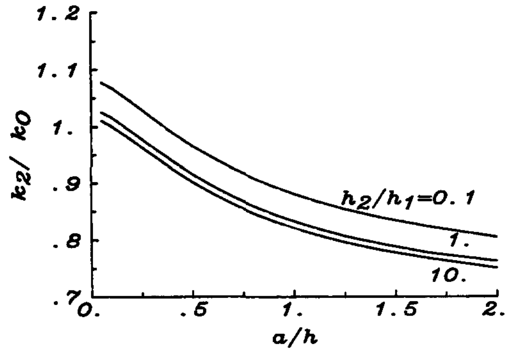

Fig. 9. The effect of h_2/h_1 and a/h on k_1/k_0 in a two-layer plate under bending. Material 1: B, Material 2: A, $k_1 = k_1(h-c_0)$, $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $h = h_1+h_2$.

shows the distribution of the stress intensity factor $k_1(\bar{z})$, $(\bar{z} = z+c_0)$ along the crack front for a two-layer plate under membrane loading. Note that the Young's modulus E_x in the loading direction in layer 1 is greater than that in layer 2 (Material A and Material B in Table 5). Consequently, $k(\bar{z})$ in layer 1 is greater than $k(\bar{z})$ in layer 2. This, of course, is the physically expected result.

In the third group of examples we consider the crack problem in a plate that consists of three dissimilar layers [Fig. 1(c)]. Since there is again no material symmetry, the membrane and bending components of the solution would always be coupled regardless of loading. Some calculated results showing the thickness distribution of the stress intensity factor $k_1(\bar{z})$, $(\bar{z} = z+c_0)$ along the crack front are given in Fig. 11.

The remaining results given in this section concern the rather peculiar role the Poisson's ratio plays in the thickness distribution of the stress intensity factor in bonded isotropic layers under membrane loading. For a plate which consists of two bonded isotropic layers the effect of the Poisson's ratio is shown in Figs 12 and 13, where $E_2/E_1 = 5$, $\nu_1 = 0.3$ and ν_2 is varied between 0 and 0.45. The interesting result here is that under neutral plane membrane loading the normalized stress intensity factor turns out to be independent of the crack length if the two layers have the same Poisson's ratio. This seems to be the consequence of a more general result, namely that in laminated plates which consist of arbitrary number of dissimilar isotropic layers, membrane and bending solutions uncouple if all layers have the same Poisson's ratio. The result follows directly from the fact that in this case the

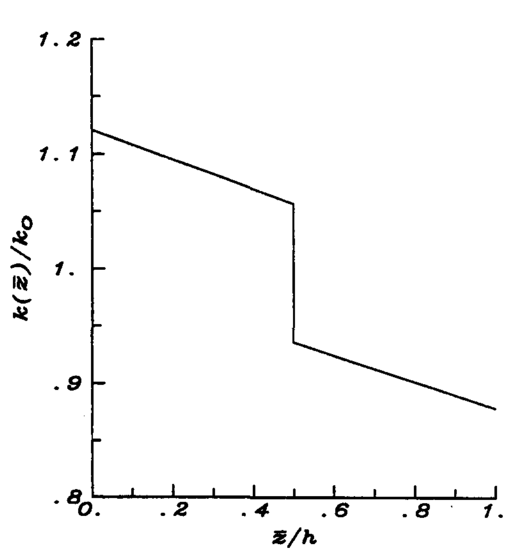

Fig. 10. Thickness distribution of the normalized stress intensity factor $k(\bar{z})/k_0$ in a two-layer plate under membrane loading $N_{xx} = N_0$. Layer 1 is Material A, Layer 2 is Material B, $\bar{z} = z+c_0$, $k_0 = \sigma_t\sqrt{a}$, $\sigma_t = N_0/h$, $a/h = 1$, $h_1/h_2 = 1$.

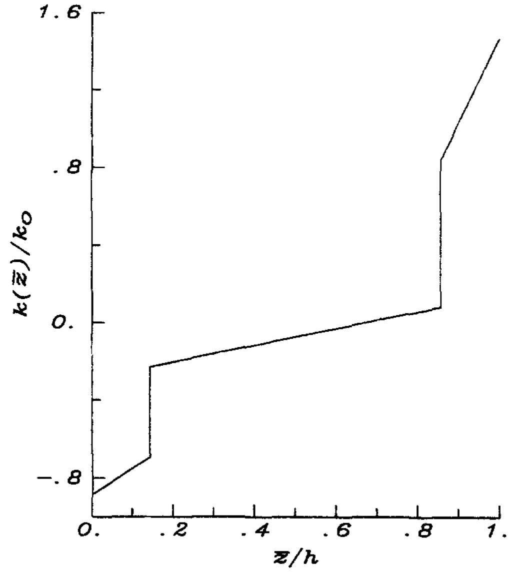

Fig. 11. The thickness distribution of the normalized stress intensity factor $k_1(\bar{z})/k_0$ along the crack front in a plate that consists of three dissimilar layers and is subjected to cylindrical bending, $M_{xx} = M_0$. The surface layers, layers 1 and 3 are isotropic with the elastic constants $3E$, $\nu = 0.3$ and $10E$, $\nu = 0.3$, respectively. Layer 2 has the constants $E_i = E$, $\nu_{ij} = 0.3$ $(i, j = x, y, z)$, $G_{xy} = E/2(1+0.3)$, $G_{xz} = G_{yz} = 3G_{xy}$; $\bar{z} = z + c_0$, $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $a/h = 0.5$, $h_3 = h_1 = 0.2h_2$ [Fig. 1(c)].

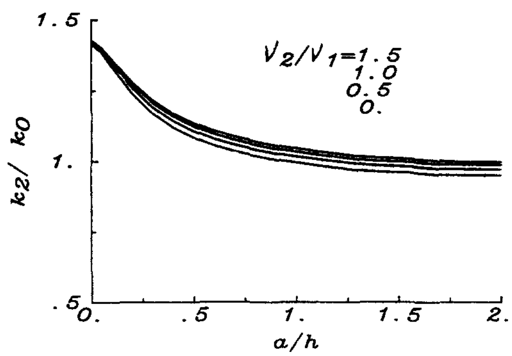

Fig. 12. The effect of the Poisson's ratio on the normalized stress intensity factor k_2/k_0 in a plate which consists of two isotropic layers and is subjected to bending $M_{xx} = M_0$ [Fig. 1(b)]. Layers 1 and 2 are isotropic with $E_2/E_1 = 5$, $\nu_1 = 0.3$, ν_2 variable, $k_2 = k_1(h - c_0)$, $k_0 = \sigma_b\sqrt{a}$, $\sigma_b = 6M_0/h^2$, $h_1/h_2 = 1$.

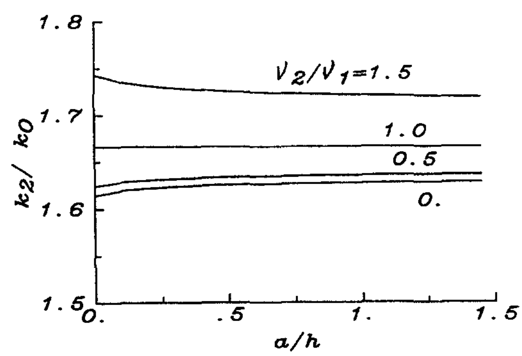

Fig. 13. Same as Fig. 12, the plate is under membrane loading $N_{xx} = N_0$, $k_0 = \sigma_t\sqrt{a}$, $\sigma_t = N_0/h$.

coefficients B_{ij} $(i, j = 1, 2)$, and B_{66} of the plate stiffness matrix become zero [see eqns (9)–(13) and the discussion that follows]. For example, for a two layer medium from Fig. 1(b) and (12) it is seen that

$$\begin{bmatrix} B_{ij} \\ B_{66} \end{bmatrix} = \int_{-c_0}^{h_1-c_0} \begin{bmatrix} C_{ij}^1 \\ C_{66}^1 \end{bmatrix} z\,dz + \int_{h_1-c_0}^{h-c_0} \begin{bmatrix} C_{ij}^2 \\ C_{66}^2 \end{bmatrix} z\,dz$$

$$= \begin{bmatrix} C_{ij}^1 \\ C_{66}^1 \end{bmatrix} [(h_1-c_0)^2 - c_0^2] + \begin{bmatrix} C_{ij}^2 \\ C_{66}^2 \end{bmatrix} [(h-c_0)^2 - (h_1-c_0)^2], \quad (i, j = 1, 2). \quad (55)$$

For an isotropic material we have [see (6), (7) and (17)]

$$C_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad C_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad C_{66} = \frac{E}{2(1+\nu)}, \quad (56)$$

$$\bar{C}_{11} = \frac{E}{(1-\nu)(1+\nu)}, \quad \bar{C}_{12} = \frac{E\nu}{(1-\nu)(1+\nu)}. \quad (57)$$

From (55), (56) and (57) it then follows that B_{11}, B_{12} and B_{66} (defined in terms of either C_{ij} *or* $\bar{C}_{ij}$) would vanish if the two layers have the same Poisson's ratio and if the distance c_0 defining the neutral plane is calculated from

$$c_0 = h_1 \frac{\frac{1}{2} + \frac{E_2}{E_1}\left[\frac{h_2}{h_1} + \frac{1}{2}\left(\frac{h_2}{h_1}\right)^2\right]}{1 + \frac{E_2}{E_1}\frac{h_2}{h_1}}, \quad (58)$$

where E_1 and E_2 are the Young's moduli of the layers 1 and 2, respectively. In a layered plate it may easily be shown that c_0 given by (58) is the same as the location of the neutral plane one obtains from elementary equilibrium considerations. One can also show that the result $B_{ij} = 0 = B_{66}$ $(i, j = 1, 2)$, remains valid for any number of layers having only in-plane isotropy and equal in-plane Poisson's ratios, and the stress intensity factors are independent of the actual value of the common Poisson's ratio.

In the uncoupled case the membrane solution of a laminated plate containing a through crack may be approximated by the solution of the corresponding generalized plane stress problem. Observing that in this case the crack opening displacement is defined by $u_0(0, y)$ only, the stress intensity factor for the ith layer may be obtained from

$$k_1^i(a) = -\lim_{y\to a} \frac{E_i}{2}\sqrt{2(a-y)}\,g_1(y), \quad (59)$$

where $g_1(y) = (\partial/\partial y)u_0\,(0, y)$ is obtained from

$$\frac{1}{\pi}\int_{-a}^{a} \frac{g_1(t)}{t-y}\,dt = \frac{2}{E} f_1(y), \quad -a < y < a, \quad (60)$$

f_1 is defined by (33) and E is the average modulus given by

$$E = \frac{1}{h}\sum E_i h_i. \quad (61)$$

For example, for a laminate under uniform membrane loading $N_{xx} = N_0$, for the ith layer we find

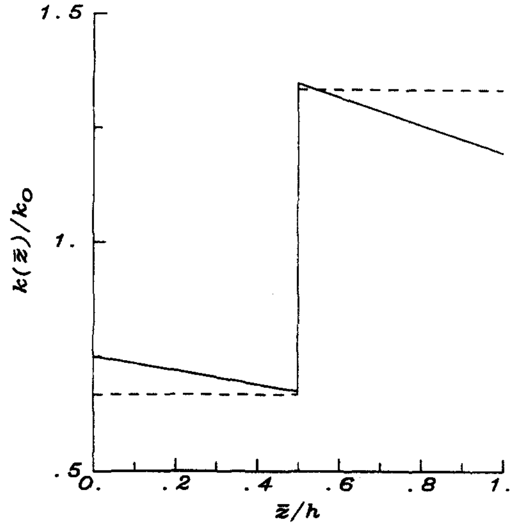

Fig. 14. Thickness variation of the stress intensity factor in a plate which consists of two isotropic layers and is subjected to membrane loading $N_{xx} = N_0$. $\bar{z} = z + c_0$, $k_0 = (N_0/h)\sqrt{a}$, $h_1 = h_2$, $E_2/E_1 = 2$, full line: $\nu_1 = 0.45$, $\nu_2 = 0.1$, dashed line: $\nu_1 = \nu_2$.

$$k_1^i(a) = \frac{E_i}{E}\frac{N_0}{h}\sqrt{a}. \tag{62}$$

For $\nu_2/\nu_1 = 1$ in the example considered in Fig. 13, (62) gives $k_2/k_0 = 5/3$ which is the same as that obtained from the integral equations described in Section 3.

The effect of differences in the Poisson's ratio of the layers on the stress intensitiy factor in a two layer laminate under membrane loading may be seen in Fig. 14. One of the examples shown by the figure has equal Poisson's ratios and, hence, piecewise constant stress intensity factor [$k_1/k_0 = 2/3$ and 4/3, see (62)]. In the other example $\nu_1 \neq \nu_2$ and, consequently, the problem is coupled, that is the membrane loading gives rise to both membrane and bending stresses. A similar example for a three layer laminate under membrane loading is shown in Fig. 15, where again the dashed lines are obtained from the integral equations given in

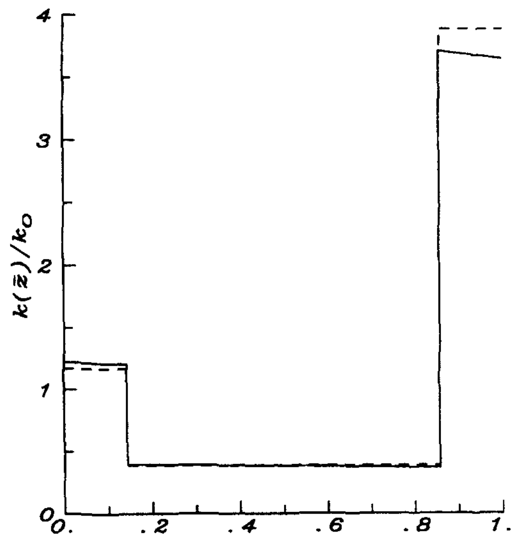

Fig. 15. Thickness variation of the stress intensity factor in a three-layer laminate under membrane loading $N_{xx} = N_0$. $\bar{z} = z + c_0$, $h_2/h_1 = 5$, $h_3 = h_1$, $k_0 = (N_0/h)\sqrt{a}$; full line: $E_1/E_2 = 3$, $E_3/E_2 = 10$, $\nu_1 = 0.5$, $\nu_3 = 0.2$, layer 2 has only in-plane isotropy with $\nu_2 = 0$, $G_{2xy} = E_2/2$, $G_{2xz} = G_{2yz} = 3G_{2xy}$; dashed line: $\nu_1 = \nu_2 = \nu_3 = 0$, $E_1/E_2 = 3$, $E_3/E_2 = 10$, $G_{2xy} = E_2/2$, $G_{2xz} = G_{2yz} = 3G_{2xy}$.

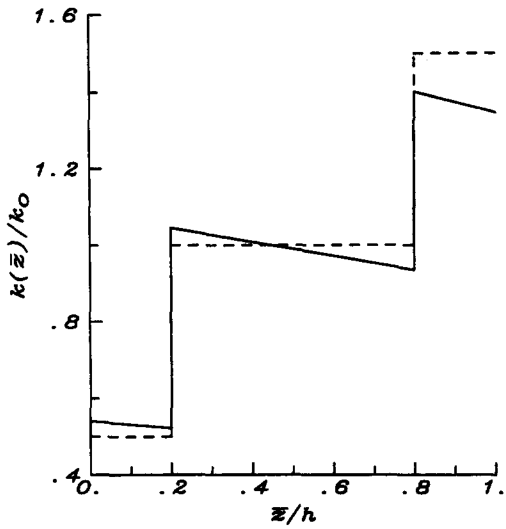

Fig. 16. Stress intensity factors in three-layer laminates under membrane loading $N_{xx} = N_0$. $\bar{z} = z + c_0$, $h_1/h = 0.2$, $h_2/h = 0.6$, $h_3/h = 0.2$, $E_2/E_1 = 2$, $E_3/E_1 = 3$; full lines: $\nu_1 = 0.3$, $\nu_2 = 0.45$, $\nu_3 = 0.1$; dashed lines: $\nu_1 = \nu_2 = \nu_3$.

Section 4 and are verified by using (62). Full lines show the result obtained from the coupled problem. In Fig. 15 the dashed lines also show an example in which layer 2 has only in-plane isotropy and yet the stress intensity factors are piecewise constant and are the same as the plane stress results given by (62). Figure 16 shows the results of another example for three-layer laminates.

The results given in this section regarding the stress intensity factors in laminated plates with a through crack indicate that generally the problem is coupled and, depending on the accuracy required, even under membrane loading, it may not be possible to avoid solving the related integral equations.

Acknowledgements—This research was supported by NSF under the Grant MSS-8917867 and by NASA-Langley under the Grant NAG-1-713.

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